Calculus II: For Physical Sciences and Engineering (AS.110.109)

Lecture: 13

Lecturer: Hunter Stufflebeam

Textbook Sections: 7.8

Todays Topic: Improper integrals

Goal: Integrate functions which:

(i) Live on infinite intervals

(ii) Have asymptotes / more serious discontinuities.

A key application of these ideas: Probability

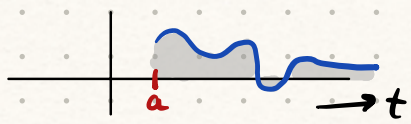
Section I: "Type 1 improper integrals"

Setup: $f: (a, \infty) \to \mathbb{R}$ or $g: (-\infty, b) \to \mathbb{R}$

Definition I:

(i) If $\int_a^t f(x)\,dx$ exists $\forall t \geq a$, then we define

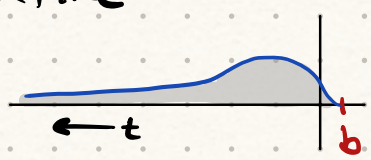
$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$



provided that the limit exists and is finite (convergence).

(ii) If $\int_t^b g(x)\,dx$ exists $\forall t \leq b$, then we define

$$\int_{-\infty}^b g(x)\,dx = \lim_{t\to-\infty} \int_t^b g(x)\,dx$$

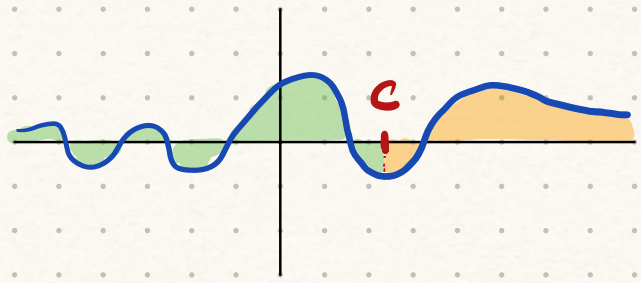


provided that the limit exists and is finite (convergence).

(iii) If $f:(-\infty,\infty)\to\mathbb{R}$ and for some $c\in\mathbb{R}$ both of $\int_{-\infty}^{c} f(x)\,dx$ and $\int_{c}^{\infty} f(x)\,dx$ are convergent, then

$$\int_{-\infty}^{\infty} f(x)\,dx := \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx.$$

In this case, we say **f(x) is integrable**.



Motto: A region might go off to infinity, even though it has finite net area.

Examples:

1) $f(x)=\frac{1}{x^2}$ on $[1,\infty)$

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}dx = \lim_{t\to\infty}\left(-\frac{1}{x}\right)\Big|_1^t = \lim_{t\to\infty}\left(1-\frac{1}{t}\right)=1$$

2) $f(x)=\frac{1}{1+x^2}$ on $(-\infty,\infty)$

$$\int_{-\infty}^{0}\frac{1}{1+x^2}dx = \int_0^\infty \frac{1}{1+x^2}dx = \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}dx = \lim_{t\to\infty}\arctan x\Big|_0^t$$

$$= \lim_{t\to\infty}\arctan(t) = \pi/2.$$

Thus $\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx=\pi$.

3) How about $f(x)=\frac{1}{x}$ on $(-\infty,-1]$?

$$\int_t^{-1}\frac{1}{x}dx = -\ln|t|, \text{ so } \lim_{t\to-\infty}\int_t^{-1}\frac{1}{x}dx=-\infty$$

So this integral **diverges**. In particular, $\frac{1}{x}$ doesn't "go to 0 fast enough at $-\infty$." Compare this with $\frac{1}{x^2}$!

4) Exercise:

Show $\int_1^\infty \frac{1}{x^p}\,dx$ converges iff $p>1$, and diverges if $p\le 1$.

Often, we pair a result like this with:

A useful Theorem: (Comparison Test for Improper Integrals)

Suppose $f, g : [a,\infty) \to \mathbb{R}$ are continuous, with $f(x) \ge g(x) \ge 0$ for all $x \ge a$.

1) If $\int_a^\infty f(x)\,dx$ converges, then so does $\int_a^\infty g(x)\,dx$

2) If $\int_a^\infty g(x)\,dx$ diverges, then so does $\int_a^\infty f(x)\,dx$.

⚠ You can easily adapt this to the cases $f, g : (-\infty, a] \to \mathbb{R}$ and $f, g : (-\infty, \infty) \to \mathbb{R}$.

Example: Converge of diverge?

5) $\int_1^\infty \frac{x+1}{\sqrt{x^4+x}}\,dx$ (notice $\frac{x+1}{\sqrt{x^4+x}} \ge \frac{x+1}{\sqrt{2x^4}} = \frac{1}{\sqrt{2}}\cdot\left(\frac{1}{x}+\frac{1}{x^2}\right)$, $x \ge 1 \Rightarrow x^4 \ge x$)

$\Rightarrow$ diverges

Challenge:

6) $\int_0^\infty e^{-x^2}\,dx$. (notice that $\forall x \ge 1$, $e^{x^2} \le e^{-x}$).

7) What about $\int_{-1}^1 \frac{1}{x}\,dx$?

Naive calculation: $\int_{-1}^1 \frac{1}{x}\,dx = \ln|x|\Big|_{-1}^1 = 0 - 0 = 0$



Question: Is this valid? Where was the error made?

Section II: "Type 2 improper integrals"

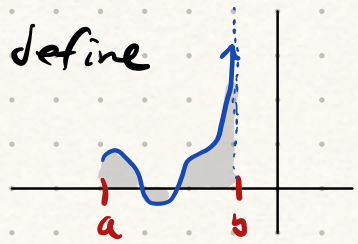
Our error in 7) lies in the fact that $\frac{1}{x}$ has a discontinuity at $x=0$, so that the FTC does not apply. Because of examples like this, we introduce a new type of improper integral as follows:

Setup: Let $f:[a,b)\to\mathbb{R}$ be cts, with a discontinuity at $x=b$.
Let $g:(a,b]\to\mathbb{R}$ be cts, with a discontinuity at $x=a$

Definition II:

(i) If $\int_a^t f(x)\,dx$ exists $\forall a\le t<b$, then we define

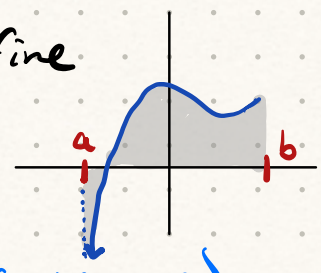
$$\int_a^b f(x)\,dx = \lim_{t\nearrow b}\int_a^t f(x)\,dx$$



provided that the limit exists and is finite (convergence).

(ii) If $\int_t^b g(x)\,dx$ exists $\forall a<t\le b$, then we define

$$\int_a^b g(x)\,dx = \lim_{t\searrow a}\int_t^b g(x)\,dx$$



provided that the limit exists and is finite (convergence).

(iii) If $f:(a,b)\setminus\{c\}\to\mathbb{R}$ is cts and for some $c\in\mathbb{R}$ (a pt. of discontinuity) both of $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then

$$\int_a^b f(x)\,dx := \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Examples

1) $\int_1^4 \frac{1}{(x-2)^2}\,dx$: $\int_1^2 \frac{1}{(x-2)^2}\,dx + \int_2^4 \frac{1}{(x-2)^2}\,dx = \lim_{t\nearrow 2}\int_1^t + \lim_{t\searrow 2}\int_t^4 = \infty$

2) **Exercise:** Show $\int_0^1 \frac{1}{x^p}\,dx$ converges iff $p<1$, and diverges if $p\ge 1$.

3) **Challenge:**

$$\int_0^1 \ln x\,dx = \lim_{t\searrow 0}\int_t^1 \ln x\,dx = \lim_{t\searrow 0}\left[x\ln x - x\right]\Big|_t^1$$
$$= -1 - \lim_{t\searrow 0}(t\ln t - t)$$
$$= -1 - \lim_{t\to 0} t\ln t$$
$$= -1$$

since $\lim_{t\to 0} t\ln t = \lim_{t\to 0}\frac{\ln t}{1/t} \overset{\text{L'Hôpital}}{=} \lim_{t\to 0}\frac{1/t}{1/t^2} = \lim_{t\to 0} t = 0$.